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THE PLANE PROBLEM OF THE IMPACT OF A PLATE ON A LIQUID STRIP OF RECTANGULAR CROSS-SECTION*

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Solutions of a number of plane problems on the impact of a plate on a finite strip of ideal incompressible fluid are obtained by using the theory of doubly-periodic functions of a complex variable. Two kinds of boundary conditions corresponding to impact of a fluid in a rectangular vessel and impact on a free layer are considered in detail. Formulas to determine the apparent mass, that are convenient for calculating modifications of known solutions /1, 2/, are obtained in a problem with a boundary condition of the first kind in the case of stiff plate impact. The dependence of the plate apparent mass on the vessel size is investigated. A comparison is given between this dependence and the results in /1-6/. The influence of the vessel shape on the apparent mass is estimated. The velocity function is determined for a supporting layer and a layer placed between walls under the impact of the plate.

The expressions obtained for the velocity function can be utilized in solving hydroelasticity problems on the dynamic interaction between a structure and a fluid, the approximate analysis of the initial stage in the puncture of obstacles by impactors when just the inertial properties of the colliding materials are taken into account, and for the test verification of numerical and other approximate methods of solving plane hydromechanics problems.

1. The impact of a plate on an incompressible fluid in a rectangular vessel. The formulation and solution of this problem were given first in /1/ by using Weierstrass sigma and zeta functions. These functions are not very convenient for computer calculations. To avoid complex transformations of the solution and to obtain more modern calculation formulas it is useful to examine the problem once again.

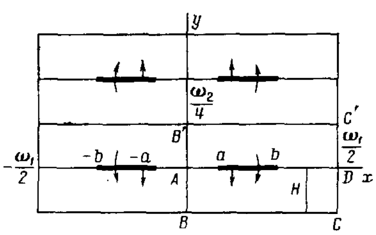

Fig.1

Following /1/, we recall its formulation to eliminate the misprint that slipped in there at the same time. We consider the impact when a flat plate in incident on the horizontal surface of a fluid at rest within a rectangular cylindrical vessel $ABCD$ (Fig.1). The plate is horizontal with the fluid at the time of making contact. To determine the perturbed motion at the time following impact, we place the origin at the left edge of the vessel at the point A, directing the x axis to the right along the free surface and the y axis perpendicularly upward. The fluid depth is denoted by $H = -i\omega_2/4$, where ω_2 is a pure imaginary quantity, the vessel width is denoted by $\omega_1/2$, while a and b are the abscissae of the plate edges at the time of impact.

To determine the characteristic function $w = \varphi + i\psi$ of the perturbed motion we have the following boundary conditions. The vertical fluid velocity component $v(x)$ is known on the segment ab. The velocity is directed downward and equals the vertical velocity of points of the plate after impact. The vessel walls AB, BC and CD are impermeable, and the normal fluid velocity thereon is zero. The impulsive pressure is zero on the free surface, i.e., $\varphi = 0$.

Continuing the flow upward through the free surface, we arrive at the problem of determining the fluid flow in the rectangle $BB'C'C$ caused by the motion of the plate ab. The sides of the rectangle are streamlines and consequently, the fluid flow can be continued through the sides of the rectangle over the whole plane. Performing the continuation we obtain a doubly-periodic fluid flow outside a lattice with periods ω_1 and ω_2. Each of the two plates, adjacent along the horizontal, have identical vertical velocities at symmetrical points while plates adjacent along the vertical have opposite vertical velocities.

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It follows from the condition $\varphi = 0$ on Aa and Db that the circulation around each plate should be zero.

The flow pattern in Fig.1 also enables us to give the problem another mechanical interpretation. For instance, it can be considered that it consists of investigating the fluid flow in a rectangular vessel of twice the width ω_1 with a thin impermeable baffle in the middle. The impact of two plates symmetrically about to the middle of the vessel is superimposed on the fluid free surface. The baffle does not influence the flow that originates and can be removed if necessary. If $a = 0$, then both plates merge and we have the symmetric impact of one plate of width $2b$.

The solution of the problem was found by using the effective general formulas (/1/, Ch. III, Sect.9) for the velocity function $dw/dz = u(x, y) - iv(x, y)$, $z = x + iy$. These formulas were first simplified and the sigma function was replaced by the Jacobi theta functions. The symmetry of the continued flow as used substantially. As a result of the transformations it turned out to be possible to express dw/dz as a definite integral satisfying all the conditions formulated above.

An analogous formula was also found for the velocity function dw/dz of a fluid flow without circulation in a rectangle of periods, which corresponded to the impact of n plates on the surface AD with the abscissae a_k and b_k $(k = 1,2,\ldots, n)$ for the left and right edges of each plate, respectively, and the normal impact velocity $v(x)$.

The formulas obtained (the definite integrals) for dw/dz contain no undetermined constants for any plate arrangement on the fluid surface. They are convenient for calculating different fluid flow elements, particularly the streamlines, the characteristic function $w = \varphi + i\psi$ etc. The necessary separation of the integrals into real and imaginary parts is realized by using known theta function addition formulas /7/.

In the general case the integrals are taken numerically.

To describe the possibilities of the proposed method of investigating impact more completely we present a formula for calculating the apparent mass of a plate of width $2b$ in a vessel of finite width $\omega_1 = 2d$ and depth $H = -i\omega_2/4$. The impact velocity is $v(x) = v_0 = \text{const}$, and the plate is arranged symmetrically about the middle of the vessel. We have for the coefficient of apparent mass

$$\lambda = 2\rho \int_0^b \frac{u(x, 0)}{v_0} x\, dx \tag{1.1}$$

where ρ is the fluid density, and $u(x, 0)$ is the horizontal component of the fluid velocity directly under the plate. Extracting $u(x, 0)$ from the expression for dw/dz for $y = 0$ and substituting into (1.1), we obtain after going over to dimensionless quantities and performing other necessary reduction

$$\bar{\lambda} = \frac{8\gamma}{\pi^2} \frac{\vartheta_1'(0)}{\vartheta_1(\gamma)} \iint_\Omega [\xi_1 (f(1, -1) g_1^+(\xi_1) + f(-1, 1) g_1^-(\xi_1)) + \tag{1.2}$$

$$\eta_1 (f(1,1) g_1^+(\eta_1) + f(-1, -1) g_1^-(\eta_1))]\, dx\, dy, \quad \bar{\lambda} = \lambda/\lambda_\infty$$

$$\gamma = b/d,\ \lambda_\infty = \pi\rho b^2/2,\ \vartheta_1'(0) = \pi\vartheta_2(0)\,\vartheta_3(0)\,\vartheta_0(0)$$

$$f(m, n) = (4\xi_2\eta_2 g^m(\xi_2)\, g^n(\eta_2))^{1/2}$$

$$\xi_1 = xy, \quad \eta_1 = 1 - \frac{x^2 + y^2}{2}, \quad \xi_2 = \frac{(x - y)^2}{4}, \quad \eta_2 = \frac{(x + y)^2}{4}$$

$$g_1^\pm(x) = \frac{\vartheta_1(\gamma(1 \mp x))}{\vartheta_1(\gamma x)} \pm \frac{\vartheta_0(\gamma(1 \mp x))}{\vartheta_0(\gamma x)}, \quad g(x) = \prod_{m=0}^{1} \frac{\vartheta_m(\gamma(1 - x))}{\vartheta_m(\gamma x)}$$

The theta functions in (1.2) are defined by the following rapidly converging series /7/:

$$\vartheta_1(x) = 2q^{1/4}(\sin \pi x - q^2 \sin 3\pi x + \ldots)$$
$$\vartheta_2(x) = 2q^{1/4}(\cos \pi x + q^2 \cos 3\pi x + \ldots)$$
$$\vartheta_3(x) = 1 + 2q \cos 2\pi x + 2q^4 \cos 4\pi x + \ldots$$
$$\vartheta_0(x) = 1 - 2q \cos 2\pi x + 2q^4 \cos 4\pi x - \ldots$$

in which it is necessary to set $q = \exp(-2\pi H/d)$ for the problem under consideration. The domain of integration Ω is a triangle $y \leqslant x \leqslant \sqrt{2} - y$, $0 \leqslant y \leqslant \sqrt{1/2}$ in the auxiliary plane xy.

Verification shows that the integrand (1.2) has a finite limit as $x \to y$ and $y \to 0$. It is continuous at other points of the triangle Ω. Consequently, despite its apparent complex form the integral (1.2) is easily evaluated by the Simpson method on the basis of standard procedures.

It is useful to compare computations of $\bar{\lambda}$ by means of (1.2) with results known in the literature /1, 2/ for the limit cases $H \to \infty$, d finite and $d \to \infty$, H finite. The former (the vessel depth is infinite but the width is finite) corresponds to the impact of a periodic lattice of plates on a half-space. The double magnitude of the apparent mass from the computation on one plate is determined according to /1/ from the formula

$$\bar{\lambda} = \frac{8}{\pi^2\gamma^2} \ln \sec \frac{\pi}{2}\gamma \tag{1.3}$$

The latter case (the vessel width is infinite, while the depth is finite) was first investigated in /2/. The plate apparent mass was determined /2/ by using multiple integrals. However, the numerical results of /2/ can be refined somewhat (and the results published later in /3/ at the same time) if the following formula is used ($\mathbf{K}$ and $\mathbf{E}$ are complete elliptic integrals of the first and second kinds):

$$\bar{\lambda} = \frac{4}{\pi^2}\frac{\alpha}{k}\int\limits_0^1 \frac{(\mathbf{K}(k) - \mathbf{E}(k)) f(x) + E(k)}{[1 - f(x)]^{3/2}} (1 - x^2)\, dx \tag{1.4}$$

$$\alpha = \pi b/(2H),\ k = \operatorname{th}\alpha,\ f(x) = \operatorname{sh}^2 \alpha x/\operatorname{sh}^2\alpha$$

It is still necessary to make a change in the variable of integration $x = 1 - t^2$, $0 \leqslant t \leqslant 1$, in (1.4). The appropriate multiple integral in /2/ was reduced to such an integral in order to increase the accuracy of calculating the plate apparent mass. We note that (1.4) also follows from the integral (1.2) but a unimodular transformation of the periods of the theta functions /8/ and additional awkward calculations must be used for this.

2. Plate impact on a free layer.

It was remarked in /9/ that the correctness of the results of computing the motion elements in problems of dynamic interaction between plates and a fluid of bounded volume can be set up by using the symmetry property. According to this property, the half-sum of any flow characteristics (the plate apparent masses, say) during impact on a free fluid and on a layer whose boundary is impermeable (fixed), should agree to a high degree of accuracy with the appropriate characteristic of plate impact on a half-space for a layer of considerable thickness. Application of this property in practice enables rough errors in the results of solving the problem to be avoided.

Consequently, in addition to problems on the impact of a plate on a fluid in a rectangular vessel, it is also useful to examine impact on a rectangular layer of fluid with a free surface when there are no walls and bottom constraining the fluid motion.

We show in Fig.2 a free ideal incompressible fluid layer $ABCD$ of thickness $H = -i\omega_2/2$ and width $2d = \omega_1/2$. We consider the plate impact to be symmetrical about the middle of the face AD. As before, the plate width is denoted by $2b$ and the origin of coordinates is placed at the centre of the plate.

The formulation of the problem agrees completely with that presented in Sect.1, with the exception that the faces AB, BC and CD of the layer are free and the impulsive pressure thereon is zero, i.e., $\varphi = 0$.

Continuing the flow through the free surface upward and to the right, we obtain a doubly-periodic fluid flow outside a lattice with periods ω_1 and ω_2. The plates adjacent along the horizontal have opposing vertical velocities at symmetric points.

We will seek the solution of the problem as in Sect.1. We obtain for the velocity function

$$\frac{dw}{dz} = A\int\limits_{-b}^{b} v(\xi)\, g_2(\xi, z)\, d\xi,\quad c = \frac{2b}{\omega_1} \tag{2.1}$$

$$g_2(\xi, z) = \chi(s)\frac{g_3(z)}{|g_3(\xi)|} - \chi(t)\frac{|g_3(\xi)|}{g_3(z)}$$

$$\chi(x) = \sum_{m=1}^{2}(-1)^{m+1}\frac{\vartheta_m(x)}{\vartheta_m(p)},\quad g_3(z) = \left(\frac{h(z)}{h(-z)}\right)^{1/2}$$

$$h(z) = \prod_{m=1}^{2}\vartheta_m\left(\frac{b+z}{\omega_1}\right),\quad A = \frac{1}{2\pi}\frac{\vartheta_1'(0)}{\omega_1\vartheta_1(c)}$$

$$p = (\xi - z)/\omega_1,\ s = p + c,\ t = p - c$$

In this case $q = \exp(-0.5\pi H/d)$ is the theta function parameter.

Different special cases can be obtained from (2.1). For instance, let the layer have infinite width $(d \to \infty)$ and finite thickness H. If the impact is realized by a stiff plate, where $v(x) = v_0 = \text{const}$ after the impact, then the integral (2.1) can be expressed in terms of elementary functions. We obtain

$$\frac{u(x, 0)}{v_0} = \operatorname{sh}\frac{\alpha x}{b}\left[\operatorname{sh}^2\alpha - \operatorname{sh}^2\frac{\alpha x}{b}\right]^{-1/2} \tag{2.2}$$

$$\alpha = \frac{\pi b}{2H},\quad -b < x < b$$

for the horizontal component of the fluid velocity $u(x, 0)$ directly under the plate.

The plate apparent mass coefficient is also easily found

$$\lambda = \frac{4}{\pi\alpha}\left(\frac{\pi}{2} - \int_0^1 \arcsin\frac{\operatorname{ch}\alpha x}{\operatorname{ch}\alpha}\,dx\right) \tag{2.3}$$

This formula is convenient for computer calculations. If the fluid layer is quite thin $(H \to 0, \alpha \to \infty)$, we obtain $\lambda = 2\rho bH$ from (2.3). For impact on a layer of large thickness $(H \to \infty, \alpha \to 0)$ we have

$$\bar{\lambda} = 1 - \alpha^2/6 \tag{2.4}$$

Taking into account that upon impact on a fluid of finite depth /2/ (a fixed layer) we obtain from (1.4) as $H \to \infty$

$$\bar{\lambda} = 1 + \alpha^2/12 \tag{2.5}$$

It can be said that the half-sum of the apparent masses (2.4) and (2.5) for a free and a fixed fluid layer is close to the apparent mass of a plate upon impact on a half-space.

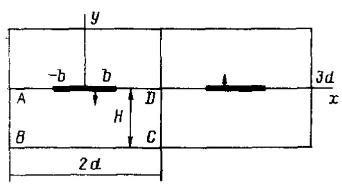

Fig.2

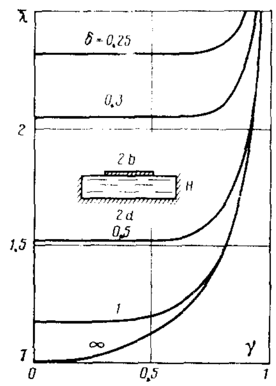

Fig.3

3. Impact on a supported layer and on a layer placed between walls. We will modify the boundary conditions by assuming that along the face BC in Fig.2 an impermeable wall is located on which the normal fluid velocity equals zero. As before, the faces AB and CD are considered free and $\varphi = 0$ thereon.

Continuing the flow we arrive at the definition of a doubly-periodic flow having the fundamental periods $\omega_1 = 4d$ and $\omega_2 = 4iH$. The complex velocity function is defined by (2.1) in which we must put

$$c = b/d, \quad q = \exp(-\pi H/d) \tag{3.1}$$

$$\chi(x) = \sum_{m=0}^{3} (-1)^{m+1}\frac{\vartheta_m(x)}{\vartheta_m(p)}, \quad h(z) = \vartheta_1\left(2\frac{b+z}{\omega_1}\right)$$

The definition of the other quantities is given in (2.1).

Formulas for doubling the theta functions /8/ were used in deriving (3.1). This solution is conserved if a thin impermeable baffle is placed along the y axis in Fig.2. We obtain a new mechanical interpretation of the problem - the motion of a plate submerged in the middle of a rectangular vessel and located perpendicular to the fluid free surface is investigated.

The solution (3.1) extends the results obtained in /9, 10/.

Now let impermeable walls be located along the faces AB and CD (Fig.2) while the face BC is free and $\varphi = 0$ thereon. The continued flow will have the fundamental periods $\omega_1 = 2d$ and $\omega_2 = 2iH$. As before, the velocity function dw/dz is determined by the integral (2.1) in which it is necessary to take

$$c = \frac{b}{2d}, \quad q = \exp\left(-\frac{\pi H}{d}\right) \tag{3.2}$$

$$\chi(x) = \frac{\vartheta_1(x)}{\vartheta_1(p)}, \quad h(z) = \vartheta_1\left(\frac{b+z}{\omega_1}\right)$$

The remaining quantities in (2.1) do not change.

In all the examples considered, the function $g_3(z)$ takes real positive values on the upper edges of the slits (segments).

4. Results of calculations. Figure 3 shows graphs of the dependence of the dimensionless coefficient $\bar{\lambda}$ on the ratio γ for different values of the parameter $\delta = H/b$ that equals the ratio between the vessel depth and the plate half-width.

The lower curve corresponds to the case $\delta = \infty$ (a vessel of infinite depth) and was computed using (1.3). For $\gamma = 0$ corresponding to a problem in /2/, the quantity $\bar{\lambda}$ was calculated by using the definite integral (1.4). Below we present the results of computations using (1.4) (the last four columns determine the ordinates of the graphs in Fig.3 for $\gamma = 0$):

δ	∞	5	1	0.5	0.3	0.25
$\bar{\lambda}$	1	1.0081	1.1725	0.519	2.045	2.318

The correctness of the evaluation of the integral (1.4) on a computer was verified for $\delta \geqslant 5$ by comparing it with the results of a computation by the approximate formula (2.5) and by the asymptotic formulas in /9/ for small δ.

Note that the numerical result $\bar{\lambda}$ =1.007 /3, 4/ agrees with the value obtained for the apparent mass (AM) for $\delta = 5$. For $\delta = 1$ the value $\bar{\lambda}$=1.165 cited in the literature /1, 2, 5/ also turns out to be close to that obtained here.

The results of using (1.2) were an additional check on the accuracy of evaluating integral (1.4). Evaluation of this double integral showed that for small ratios δ the AM coefficient keeps the value practically unchanged over a broad range of variation of γ and is governed mainly by the closeness of the vessel bottom. The vessel walls have practically no influence on the impact and the flow near the plate is similar to a considerable extent to /2/.

If the plate width is close to the vessel width, then the AM grows rapidly as γ grows, mainly because of the closeness of the plate edges to the vessel walls. In the limit as $\gamma \to 1$ the plate AM are practically independent of the vessel depth, and all the graphs merge with the graph for the specific AM of a lattice of plates on a half-space.

The influence of the vessel shape on the plate AM ceases to be felt in practice for $\delta > 5$ over the whole range of variation of γ (for small δ as $\gamma \to 1$). In particular, for $\delta = 5$ the graph of $\bar{\lambda}$ almost merges with the lower curve in Fig.3.

It is also useful to compare the AM of plates on a fluid on a channel of cylindrical shape /6/ with the specific AM of a periodic lattice of plates $(\delta = \infty)$ and vessels of finite depth $(\delta > 5)$. To do this it is necessary to use the equality $\sqrt{\varkappa} = \gamma$ in the appropriate formulas /6/. It turns out that the difference in AM does not exceed 8% over the whole range of variation of the parameter γ. It is possible that it is still less; however; to obtain a better-founded estimate it would be necessary to narrow the domain of applicability somewhat for the asymptotic formula /6/ for the plate AM as $\varkappa \to 1$ $(\gamma \to 1)$:

$$\bar{\lambda} = \frac{2}{\pi^2}\left(4\ln\frac{1}{1-\gamma^4} + 16\ln 2 - 2 - \frac{\pi^2}{2}\right) \tag{4.1}$$

Indeed, according to (4.1) the plate AM is 4-8% less in the range $0.9 < \gamma < 0.95$ than for a rectangular vessel of infinite depth according to (1.3). But this should obviously not be so because of the more constricted fluid motion in a cylindrical channel. However, (4.1) yields a 0.82% greater AM than (1.3) even for γ =0.99. For values of γ still closer to one, the difference in these AM becomes almost unnoticeable, which shows that the vessel shape has only a slight influence under the conditions mentioned.

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CENTRIFUGAL WAVES IN A PROGRESSIVELY ROTATING FLUID FLOW*

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It is shown that it is possible for solitons to arise in the progressively rotating flow of an ideal incompressible fluid. Such a motion is characteristic for vortical MHD generators and small-scale atmospheric vortices.

1. Equations of motion and boundary conditions. Let a progressively rotational fluid flow be created in a rigid tube with internal radius R by means of a tangential input and pressure drop. This leads to the formation in the tube of a cylindrical cavity of radius r_0 filled with air or, if the tube does not communicate with the atmosphere, the saturated vapour of the fluid (Fig.1). Any perturbation $\eta(z,t)$ of the radius of the cavity may propagate along the axis of the tube in the form of plane waves. In future, we shall assume that the maximum amplitude of the perturbation, a, is small compared with h, the thickness of the fluid layer, and l, the length of the perturbation is, in the other hand, large compared with h. This leads to the following parameters being small: $\varepsilon = a/h$ and $\delta = h/l$. Here the thickness of the fluid layer is taken to be small compared with the radius of the tube, so that $h = (R^2 - r_0^2)/(2r_0)$.

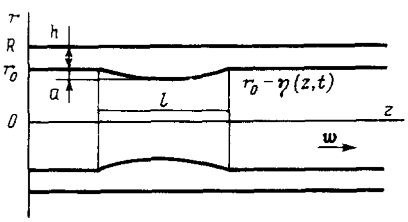

Fig.1

Because of the axial symmetry of the fluid boundary (but not of the flow), in a cylindrical coordinate system the components of $\mathbf{v}$, the velocity vector of the fluid, depend only on the distance r from the flow axis, the z coordinate and t, the time. The vorticity ω of the flow is taken to be constant along the tube and directed along the z axis, so that there is no angular dependence.

Assuming that the fluid is incompressible, we can introduce the vector potential $\mathbf{A}$ such that $\mathbf{v} = \operatorname{rot}\mathbf{A}$, and reduce the problem to solving Poisson's equation $\Delta\mathbf{A} = -\boldsymbol{\omega}$.

On the free surface of the fluid $r_1 = r_0 - \eta(z,t)$ (here and below the index 1 refers to quantities that are calculated on the free surface) the kinematic boundary condition can be written in the form

$$v_{r1} = -\left(\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial z}v_{z1}\right) \tag{1.1}$$

The dynamic boundary condition is obtained from the Euler equation by substituting expressions for the pressure in the rotating fluid at an arbitrary point of the free surface into it

$$p_1 = p_0 + {}^1/_2\rho M^2(r_0^{-2} - r_1^{-2})$$

where $M = v_\varphi r_1$ is the constant specific angular momentum of the fluid. The variation in the fluid pressure on the free surface caused by its perturbation (with $r_1 \approx r_0$) is equal to $dp_1 = \rho v_\varphi^2 r_0^{-1} d\eta$. Consequently, the radial and azimuthal projections of the Euler equation express the constant nature of the radial and azimuthal components of the fluid flow velocity on the boundary with the gas vortex, and the axial, projection gives the dynamical boundary condition:

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